The Cost of Uncertainty about the Timing of Social Security Reform*

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PRELIMINARY

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Abstract

For the Social Security program to remain solvent over the long term, either benefits must be cut or taxes must increase. Despite this well known reality, the exact timing and structure of reform remains uncertain. Thus, delaying reform not only has implications for intergenerational equity, it also creates uncertainty about the timing of reform, imposing costs on individuals as they make consumption and saving decisions. This paper assesses the welfare consequences of Social Security reform uncertainty. We find that individuals with average income bear a moderate cost from uncertainty about the timing of either tax increases or benefit cuts. Due to the progressivity of the benefit formula, low-income individuals bear a disproportionate share of the cost of uncertainty about the timing of benefit cuts. In contrast, uncertainty about the timing of tax increases is more harmful to high-income individuals.

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1. Introduction

It is well known that the Social Security Old Age, Survivors, and Disability Insurance (OASDI) program faces severe long run solvency concerns. The 2013 Social Security Trustees Report projects that the program's trust fund will run out of money in the year 2033. This means that over the coming decades either promised retirement benefits must be cut or the payroll taxes used to fund them must be increased to keep the program solvent. Gokhale (2013) estimates that an immediate tax increase of 3.1% of taxable wages, or an immediate benefit cut of 22.9%, will keep the OASI part of the program solvent for the infinite horizon.[1] While there is very litle uncertainty about the need for reform, there is a great deal of uncertainty surrounding its timing and structure. This uncertainty imposes costs on individuals as they make consumption and savings decisions in planning for their retirement.

Even though Social Security reform has been widely studied,[2] relatively little attention has been paid to the consequences of uncertainty surrounding the timing and structure of reform. Part of the reason for this gap in the literature is that the analytical tools for studying this problem are not well known. Reform represents a regime switch, after which either benefits are lower or taxes are higher. When the timing and structure of reform are uncertain, the regime switch occurs at an unknown date and the state equation after the switch is also unknown. Individual responses to Social Security reform uncertainty are best modeled using a two-stage optimal control problem. In order to study the consequences of this type of uncertainty, we provide a summary of the Maximum Principle for two-stage problems with a stochastic regime switch point and stochastic future state equation to serve as a simple template for solving such problems.[3]

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[1] If the disability component of the program is also included, then the required tax increase (assuming no behavioral response) is 4% and the required benefit cut is 23.9% as estimated in the 2013 Trustees Report.

[2] While the academic literature is too vast to do a full summary here, the consequences of most reform proposals have been studied in great depth. In addition, the Social Security Administration's Office of the Chief Actuary routinely evaluates reform proposals for their effects on Social Security's solvency. These evaluations are available at http://www.socialsecurity.gov/oact/solvency/index.html.

[3] Examples of two-stage control problems with a stochastic regime switch can be found in the early studies on resource extraction (Dasgupta and Heal (1974)) and operations research (Kamien and Schwartz (1971)). Also, problems with uncertainty about the structure of the new regime appeared in the early resource extraction literature (Hoel (1978)) and later in the technology adoption literature (Hugonnier et al. (2006), Pommeret and Schubert

After summarizing this approach, we construct a life-cycle consumption and saving model in which households face uncertainty about their lifespan, uncertainty about the timing of reform, and uncertainty about the structure of reform. The template for solving stochastic two-stage problems is then used to solve the model for the household Euler equation and simulate the optimal consumption and savings paths.

We parameterize the model to match individual wage and survival profiles. The parameterized model allows us to assess the welfare cost of uncertainty surrounding the timing and structure of Social Security reform. Because Social Security benefits and taxes vary with income, the impact of Social Security reform uncertainty may vary across income groups. To be more specific, Social Security is financed by a proportional payroll tax up to a taxable maximum wage ($113,700 in 2013). But benefits are paid according to a progressive formula that gives individuals with low lifetime earnings higher replacement rates (benefits as a share of average lifetime earnings). Since all individuals face the same tax rate (at least up through the taxable maximum income), but receive different replacement rates, we can vary the replacement rate in our model to simulate the effect of Social Security reform uncertainty for individuals of different incomes.

Uncertainty about the timing of reform is considered first because timing uncertainty is harder to deal with theoretically and because less is known about its welfare consequences. In this scenario, the individual has full information about the structure of reform and we consider the possibility of either a known benefit cut or a known tax increase occurring at an unknown date.

Uncertainty about the timing of reform has moderate costs to individuals of average income, a result which holds regardless of whether reform takes the form of across-the-board tax increases or across-the-board benefit cuts. However, tax increases and benefit cuts have different effects for low-income and high-income individuals. If individuals know that the Social Security budget will be balanced using an across-the-board benefit cut, uncertainty about the timing of the reform harms low-income individuals more because Social Security accounts for a larger portion of their lifetime wealth. Alternatively, when the budget will be balanced using an across-the-board tax increase, high-income individuals face larger welfare losses because they face the same percentage

(2009), Abel and Eberly (2012)). Our theoretical contribution is to build a bridge between these literatures and the two-stage control literature by generalizing these specific examples and summarizing how to nest both layers of uncertainty together in a control problem.

tax increase but rely less on the progressive benefit.

Of course, in addition to timing uncertainty, individuals do not know the structure of future reform either. The government may balance the budget through tax reform, benefit reform, or some combination of the two. After adding this second layer of uncertainty to our model, the overall welfare cost of the combined reform uncertainty is less than the cost for the case where there is only timing uncertainty about benefit reform but greater than the case where there is only timing uncertainty about tax reform.

This paper is most closely related to Gomes et al. (2007) and Luttmer and Samwick (2012), who also attempt to quantify the costs of Social Security reform uncertainty. Gomes et al. (2007) study a model environment in which there is uncertainty about whether a benefit cut will occur at a given future date. They find that individuals would be willing to give up 0.6 percent of their lifetime earnings in exchange for learning about the cut to Social Security benefits at age 28 instead of age 65. While this result has a similar flavor to ours, their model likely overstates the result by placing the uncertainty at the worst possible date (i.e., at retirement) rather than modeling the full uncertainty about the timing of reform.[4] Luttmer and Samwick (2012) use survey data to elicit the degree of Social Security reform uncertainty that individuals perceive, and how costly such uncertainty is to them. They find that, on average, individuals between the age of 25 and 59 expect to receive only 60% of the Social Security benefits that they have been promised. Moreover, individuals would be willing to tolerate an additional 4-6% cut in benefits in exchange for certainty about their level.

This paper is also related to the growing literature on policy uncertainty more generally. Uncertainty about future policy can affect investment, saving, and hiring decisions (Rodrik (1989), Bernanke (1983)). More recent studies suggest that the impact of policy uncertainty on firms' decisions and profitability can be substantial, potentially resulting in large efficiency costs. Several of these studies estimate or calibrate models in which taxes, government spending, or other policies are uncertain, and then explore the economic impacts of this uncertainty (Fernández-Villaverde

[4]Evans et al. (2012) consider a two-period model in which future transfers to the old are uncertain because transfers are based on the stochastic wage income of the young. If promised transfers are infeasible because of a negative productivity shock, the government switches to a more affortable transfer scheme. Households anticipate this risk, which in turn affects the equity premium in general equilibrium.

et al. (2011), Croce et al. (2012), Hassett and Metcalf (1999), Pastor and Veronesi (2011), Sialm (2006), Ulrich (2012)). Other studies use the timing of national elections as a measure of policy uncertainty, examining the impact of uncertain election outcomes on economic behavior (Belo et al. (2011), Boutchkova et al. (2012), Durnev (2010), Julio and Yook (2012), Pantzalis et al. (2000)). In this paper, we focus on a specific, understudied aspect of policy uncertainty, namely uncertainty surrounding Social Security reform.

2. Theory: Maximum Principle for Stochastic Two-Stage Control

Most of the two-stage optimal control literature has focused on the case of a deterministic switch point that may be a choice variable or exogenous (Kemp and Long (1977), Tomiyama (1985), Amit (1986), Tahvonen and Withagen (1996), Makris (2001), Boucekkine et al. (2004), Dogan et al. (2011), Saglam (2011), Boucekkine et al. (2012), Boucekkine et al. (2013a), Boucekkine et al. (2013b) and many others). However, there are examples from the early literature on resource extraction (Dasgupta and Heal (1974)), operations research (Kamien and Schwartz (1971)), and environmental catastrophe (Clarke and Reed (1994)) that solve specific problems with a stochastic regime switch. In this section we provide a self contained summary of how to solve a generic two-stage control problem with a stochastic regime switch. We also show how to augment the method to allow for uncertainty about the structure of the new regime.

Time is continuous and indexed by t. The planning interval, which begins at $t = 0$ and ends at $t = T$, is comprised of two stages. The first stage stretches from $t = 0$ to $t = t_1$, and the second stage stretches from $t = t_1$ to $t = T$. Each stage has a unique performance index and/or a unique state equation. The regime switch point t_1 is stochastic, with probability density $\phi(t_1)$ and sample space $[0, T]$. The control variable $u(t)$ is unconstrained and the state variable $x(t)$ is constrained only at the initial and terminal points in time. Other complications are avoided to focus on the task at hand. Let us state the problem formally.

Problem 1 (Two-Stage Problem with Stochastic Switch Point).

$$\max_{u(t)_{t\in[0,T]}} : J = \mathbb{E}\left[\int_0^{t_1} f_1(t, u(t), x(t))dt + \int_{t_1}^{T} f_2(t, u(t), x(t))dt\right], \tag{1}$$

subject to

$$\frac{dx(t)}{dt} = g_1(t, u(t), x(t)), \text{ for } t \in [0, t_1], \tag{2}$$

$$\frac{dx(t)}{dt} = g_2(t, u(t), x(t)), \text{ for } t \in [t_1, T], \tag{3}$$

$$x(0) = x_0, \; x(T) = x_T, \tag{4}$$

$$t_1 \text{ random with density } \phi(t_1) \text{ and sample space } [0, T]. \tag{5}$$

It is helpful to lay out some assumptions and definitions before stating the necessary conditions.

Assumption 1 (Differentiability). The functions f_1, f_2, g_1, and g_2 are continuously differentiable in their arguments t, $u(t)$, and $x(t)$. And the density function $\phi(t_1)$ is continuously differentiable.

Definition 1 (Solution Notation). Let $(u_2^*(t|t_1, x(t_1)), x_2^*(t|t_1, x(t_1)))_{t\in[t_1,T]}$ be the optimal control and state path for any t *after* the realization of the random regime switch, conditional on the switch date t_1 and conditional on the stock of the state variable at that switch date $x(t_1)$. Similarly, let $(u_1^*(t), x_1^*(t))_{t\in[0,T]}$ be the optimal control and state path for any t *before* the realization of the random switch. Hence, the path that is actually followed, conditional on switch date t_1, is $(u_1^*(t), x_1^*(t))_{t\in[0,t_1]}$ and $(u_2^*(t|t_1, x_1^*(t_1)), x_2^*(t|t_1, x_1^*(t_1)))_{t\in[t_1,T]}$.

The following theorem summarizes the solution technique for a generic two-stage control problem with a stochastic switch. We refer readers to Appendix A for a full derivation of the theorem.

Theorem 1 (Necessary Conditions to Problem 1). The necessary conditions can be derived recursively in two steps.

Step 1. Solve the post-switch ($t = t_1$) subproblem:

The program $(u_2^*(t|t_1, x(t_1)), x_2^*(t|t_1, x(t_1)))_{t\in[t_1,T]}$ solves a fixed endpoint Pontryagin subproblem

$$\max_{u(t)_{t\in[t_1,T]}} : J_2 = \int_{t_1}^{T} f_2(t, u(t), x(t))dt, \tag{6}$$

subject to

$$\frac{dx(t)}{dt} = g_2(t, u(t), x(t)), \text{ for } t \in [t_1, T], \tag{7}$$

$$t_1 \text{ given}, \ x(t_1) \text{ given}, \ x(T) = x_T. \tag{8}$$

Given the Hamiltonian function

$$\mathcal{H}_2 = f_2(t, u(t), x(t)) + \lambda_2(t)g_2(t, u(t), x(t)), \tag{9}$$

the necessary conditions that must hold on the path $(u_2^*(t|t_1, x(t_1)), x_2^*(t|t_1, x(t_1)))_{t\in[t_1,T]}$ include $\partial\mathcal{H}_2/\partial u(t) = 0$ and $d\lambda_2(t)/dt = -\partial\mathcal{H}_2/\partial x(t)$. For convenience, change the time dummy t to z, and change the switch point t_1 to t and write the solution $(u_2^*(z|t, x(t)), x_2^*(z|t, x(t)))_{z\in[t,T]}$. Thus we have the optimal control and state paths for all points in time z greater than switch point t.

Step 2. Solve the pre-switch ($t = 0$) subproblem:

The program $(u_1^*(t), x_1^*(t))_{t\in[0,T]}$ solves a fixed endpoint Pontryagin subproblem with continuation function $\mathcal{S}(t, x(t))$:

$$\max_{u(t)_{t\in[0,T]}} : J_1 = \int_0^T \left\{ \left[\int_t^T \phi(t_1)dt_1 \right] f_1(t, u(t), x(t)) + \phi(t)\mathcal{S}(t, x(t)) \right\} dt, \tag{10}$$

subject to

$$\mathcal{S}(t, x(t)) = \int_t^T f_2(z, u_2^*(z|t, x(t)), x_2^*(z|t, x(t)))dz, \tag{11}$$

$$\frac{dx(t)}{dt} = g_1(t, u(t), x(t)), \text{ for } t \in [0, T], \tag{12}$$

$$x(0) = x_0, \ x(T) = x_T. \tag{13}$$

Given the Hamiltonian function

$$\mathcal{H}_1 = \left[\int_t^T \phi(t_1)dt_1\right] f_1(t, u(t), x(t)) + \phi(t)\mathcal{S}(t, x(t)) + \lambda_1(t)g_1(t, u(t), x(t)), \tag{14}$$

the necessary conditions that must hold on the path $(u_1^*(t), x_1^*(t))_{t\in[0,T]}$ include $\partial\mathcal{H}_1/\partial u(t) = 0$ and $d\lambda_1(t)/dt = -\partial\mathcal{H}_1/\partial x(t)$.

Corollary 1 (Sufficiency). If g_1 and g_2 are linear in $u(t)$ and $x(t)$, and if the integrands of J_1 and J_2 are concave in $u(t)$ and $x(t)$, then the necessary conditions are also sufficient (Mangasarian (1966)).

Checking the concavity of the integrand of J_2 is standard. But checking the concavity of the integrand of J_1 is more involved. This is because the integrand of J_1 depends on the optimal post-switch path. Thus, one must first derive the post-switch solution $(u_2^*(z|t, x(t)), x_2^*(z|t, x(t)))_{z\in[t,T]}$, which depends on $x(t)$, and then insert this solution into $\mathcal{S}$ before checking the concavity of the integrand of J_1.

Finally, to conclude this section we note that in addition to stochastic *timing* of the regime switch, we can easily allow for the possibility that the *characteristics* of the new regime itself (the functional form of the post-switch state equation) are uncertain. Adding in this second layer of uncertainty is relatively easy and requires just a few adjustments to Problem 1 and Theorem 1.[5]

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[5] Models with uncertainty about the structure of the new regime appeared in the early resource extraction literature (Hoel (1978)) and then again in the more modern literature on technology adoption when future returns to technology are stochastic (Hugonnier et al. (2006), Pommeret and Schubert (2009), Abel and Eberly (2012)). Problem 2 and Theorem 2 show how to augment our previous control problem to include both layers of uncertainty.

Problem 2 (Two-Stage Problem with Stochastic Switch Point and Stochastic Regime). Add the following to Problem 1: the uncertainty about the functional form of g_2 is summarized by the random variable α, with density $\theta(\alpha)$ and sample space normalized to $[0, 1]$, where $\theta(\alpha)$ is continuously differentiable and realizations of α and t_1 are uncorrelated.

Theorem 2 (Necessary Conditions to Problem 2). Follow Theorem 1 to recursively derive necessary conditions with the following slight modifications: in Step 1 use the notation $g_2(t, u(t), x(t)|\alpha)$ and $(u_2^*(t|t_1, x(t_1), \alpha), x_2^*(t|t_1, x(t_1), \alpha))_{t\in[t_1,T]}$ to emphasize dependence of the solution on the realization of α, and in Step 2 write the continuation function $\mathcal{S}(t, x(t), \alpha)$ and replace the last term in the integrand of J_1 with $\int_0^1 \theta(\alpha)\phi(t)\mathcal{S}(t, x(t), \alpha)d\alpha$.

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3. Application: Uncertainty about the Timing of Reform

3.1. Notation

Age is continuous and is indexed by t. Households are born at $t = 0$ and pass away no later than $t = T$. The probability of surviving to age t is $\Psi(t)$. Retirement occurs exogenously at $t = t_R$. Labor is supplied inelastically. A given household collect wages at rate $w(t)$ during the working period.

The government's *current* policy is summarized by a tax rate on wage earnings and benefit annuity (τ_1, b_1). The current policy is unsustainable and this is publicly known. Therefore households know that reform is coming, but they don't know when. The reform date t_1 is a random variable with probability density $\phi(t_1)$ and sample space $[0, T]$.

The post-reform policy is (τ_2, b_2). For now, we assume households have full information about the nature of the reform (τ_2, b_2), they just don't know when it will kick in. Thus we are dealing with an application of Problem 1 from the previous section and Theorem 1 applies.

To compress notation let $y_1(t)$ be disposable income before the reform and let $y_2(t)$ be disposable income after the reform,

$$y_1(t) = \begin{cases} (1-\tau_1)w(t), \text{ for } t \in [0, t_R], \\ b_1, \text{ for } t \in [t_R, T], \end{cases} \tag{15}$$

$$y_2(t) = \begin{cases} (1-\tau_2)w(t), \text{ for } t \in [0, t_R], \\ b_2, \text{ for } t \in [t_R, T]. \end{cases} \tag{16}$$

Consumption is $c(t)$ and savings is $k(t)$, which earns interest at rate r.

3.2. Household Problem

Period utility is CRRA with relative risk aversion σ, and utils are discounted at the rate of time preference ρ. The household solves a dynamic stochastic control problem, taking as given the disposable income functions $y_1(t)$ and $y_2(t)$ while treating the reform date t_1 as a random variable:

$$\max_{c(t)_{t\in[0,T]}} : J = \mathbb{E}\left[\int_0^T e^{-\rho t}\Psi(t)\frac{c(t)^{1-\sigma}}{1-\sigma}dt\right], \tag{17}$$

subject to

$$\frac{dk(t)}{dt} = rk(t) + y_1(t) - c(t), \text{ for } t \in [0, t_1], \tag{18}$$

$$\frac{dk(t)}{dt} = rk(t) + y_2(t) - c(t), \text{ for } t \in [t_1, T], \tag{19}$$

$$k(0) = 0,\ k(T) = 0, \tag{20}$$

$$t_1 \text{ random with density } \phi(t_1) \text{ and sample space } [0, T]. \tag{21}$$

We refer readers to Appendix B for a step-by-step derivation of the solution to this problem. Here we report just the end results. Using Theorem 1 as our guide, we solve this problem recursively: following Step 1 we find that the post-reform (after the shock has hit) consumption path is

$$c_2^*(t|t_1, k(t_1)) = \frac{k(t_1) + \int_{t_1}^T e^{-r(v-t_1)}y_2(v)dv}{\int_{t_1}^T e^{-r(v-t_1)+(r-\rho)v/\sigma}\Psi(v)^{1/\sigma}dv}e^{(r-\rho)t/\sigma}\Psi(t)^{1/\sigma}, \text{ for } t \in [t_1, T]. \tag{22}$$

Note the dependence on the timing of the reform and on the stock of assets at the time of reform.

Then, following Step 2 and using the post-reform solution and working backwards, we find that the pre-reform solution $(c_1^*(t), k_1^*(t))_{t\in[0,T]}$ solves the following system of differential equations and boundary conditions[6]

$$\frac{dc(t)}{dt} = \left(\frac{c(t)^{\sigma+1}}{\Psi(t)}\left[\frac{k(t)+\int_t^T e^{-r(v-t)}y_2(v)dv}{\int_t^T e^{-r(v-t)+(r-\rho)v/\sigma}\Psi(v)^{1/\sigma}dv}\right]^{-\sigma} e^{(\rho-r)t} - c(t)\right)\times\left[\frac{\sigma}{\phi(t)}\int_t^T \phi(t_1)dt_1\right]^{-1} + \left[\frac{d\Psi(t)}{dt}\frac{1}{\Psi(t)} + r - \rho\right]\frac{c(t)}{\sigma}, \tag{23}$$

$$\frac{dk(t)}{dt} = rk(t) + y_1(t) - c(t), \tag{24}$$

$$k(0) = 0,\ k(T) = 0. \tag{25}$$

Finally, once we have the stage-one solution $(c_1^*(t), k_1^*(t))_{t\in[0,T]}$, we can use it obtain the explicit stage-two solution conditional on realization of reform date t_1,

$$c_2^*(t|t_1, k_1^*(t_1)) = \frac{k_1^*(t_1)+\int_{t_1}^T e^{-r(v-t_1)}y_2(v)dv}{\int_{t_1}^T e^{-r(v-t_1)+(r-\rho)v/\sigma}\Psi(v)^{1/\sigma}dv} e^{(r-\rho)t/\sigma}\Psi(t)^{1/\sigma},\ \text{for } t\in[t_1,T]. \tag{26}$$

3.3. Welfare

As a point of reference for welfare comparison, consider the case where the household faces no risk (NR). The individual is endowed at $t=0$ with expected future income and solves

$$\max_{c(t)_{t\in[0,T]}} : \int_0^T e^{-\rho t}\Psi(t)\frac{c(t)^{1-\sigma}}{1-\sigma}dt, \tag{27}$$

subject to

$$\frac{dk(t)}{dt} = rk(t) - c(t), \tag{28}$$

$$k(0) = \int_0^T \phi(t_1)Y(t_1)dt_1,\ k(T) = 0, \tag{29}$$

where

$$Y(t_1) \equiv \int_0^{t_1} e^{-rv}y_1(v)dv + \int_{t_1}^T e^{-rv}y_2(v)dv. \tag{30}$$

[6] This system must be solved numerically. We guess and interate on $c(0)$ until all the constraints are satisfied.

The solution is

$$c^{NR}(t) = \frac{\int_0^T \phi(t_1)Y(t_1)dt_1}{\int_0^T e^{-rv+(r-\rho)v/\sigma}\Psi(v)^{1/\sigma}dv} e^{(r-\rho)t/\sigma}\Psi(t)^{1/\sigma}, \text{ for } t \in [0, T]. \tag{31}$$

The welfare cost of living with reform uncertainty Δ is the solution to the following equation

$$\begin{aligned} & \int_0^T e^{-\rho t}\Psi(t)\frac{[c^{NR}(t)(1-\Delta)]^{1-\sigma}}{1-\sigma}dt \qquad (32) \\ = & \int_0^T \phi(t_1)\left(\int_0^{t_1} e^{-\rho t}\Psi(t)\frac{c_1^*(t)^{1-\sigma}}{1-\sigma}dt + \int_{t_1}^T e^{-\rho t}\Psi(t)\frac{c_2^*(t|t_1, k_1^*(t_1))^{1-\sigma}}{1-\sigma}dt\right)dt_1. \end{aligned}$$

4. Parameterization

The model is parameterized to capture individual income levels and survival probabilities over their life cycle. The parameters to be chosen are the maximum lifespan T, the survival function $\Psi(t)$, the exogenous retirement date t_R, the real return on assets r, the individual discount factor ρ, the utility preference parameter governing risk aversion σ, the age-earnings distribution $w(t)$, the probability density over reform dates $\phi(t_1)$, and policy parameters capturing tax rates and benefit levels before and after reform $\{\tau_1, \tau_2, b_1, b_2\}$.

Our survival data come from the Social Security Administration's cohort mortality tables. These tables contain the mortality assumptions underlying the intermediate projections in the 2013 Trustees Report. The mortality table for each cohort provides the number of survivors at each age $a \in \{1, 2, ..., 119\}$, starting with a cohort of 10,000 newborns. However, we truncate the mortality data at age 100, assuming that everyone who survives to age 99 dies within the next year. We assume individuals enter the labor market at age 25, giving them a 75-year potential lifespan within the model. In our baseline parameterization, we use the mortality profile for males born in 1990, who are assumed to enter the labor market in 2015. For this cohort, we construct the survival probabilities at all subsequent ages conditional on surviving to age 25.

We normalize time so that the maximum age in the model is $T = 1$. Thus $t = 0$ in the model corresponds to an age of 25, and $t = 1$ corresponds to an age of 100. Because the survival data are discrete (providing the probability of surviving to each integer age), we fit a continuous survival function that has the following form:

$$\Psi(t) = 1 - t^x. \tag{33}$$

After transforming the survival data to correspond to model time, with dates on $[0, 1]$, $x = 3.28$ provides the best fit to the data.

The fixed exogenous retirement is assumed to occur at age 65, which corresponds to $t_R = \frac{40}{75}$ in the model.[7] We assume a risk-free real interest rate of 2.9 percent per year, which is consistent with the long-run real interest rate assumed by the Social Security Trustees. In our model, this implies a value of $r = 75 * 0.029 = 2.175$. Estimates of the individual discount rate ρ vary substantially in the literature, and values of $\rho < r$ are necessary to generate a hump shaped consumption profile in the model. In the baseline model we set $\rho = 0$, although we consider other values in robustness exercises. In the baseline calibration we also set $\sigma = 3$, with other values considered for robustness.

We import the individual income profile from Gourinchas and Parker (2002) with age normalized onto model time $[0, 1]$ and the maximum income normalized to one. The continuous-time wage function is approximated by fitting a fifth-order polynomial to the discrete-time wage data:

$$w(t) = 0.697 + 1.49t - 3.41t^2 + 19.08t^3 - 59.78t^4 + 52.70t^5. \tag{34}$$

There is not much evidence about the actual distribution of possible reform dates, as this depends on the political process. Although the Social Security trust fund runs out in 2033, uncertainty about reform may extend beyond that date. For example, policy makers may adopt a temporary fix as 2033 approaches, postponing major reform even further into the future. The Health and Retirement Study (HRS), an ongoing panel survey of older Americans, regularly asks respondents to rate the chances of a cut in Social Security benefits in general within the next 10

[7] While the Social Security normal retirement age is 66 for cohorts born between 1943 and 1954, and will gradually rise to 67 for cohorts born in 1960 and later, we use 65 as the exogenous retirement date for a few reasons. First, income data from Gourinchas and Parker (2002) is only available until age 65. Second, many individuals stop working and claim an actuarially reduced Social Security benefit before the normal retirement age. Finally, this assumption can make our results easier to compare with previous research, as many prior studies specify a retirement age of 65. This assumption will also be important when setting replacement rates for individuals of different incomes. We will then use replacement rates corresponding to retirement at age 65, rather than normal retirement age.

years. In the 2010 wave of the survey, the mean subjective probability of a benefit cut within the next 10 years is around 65 percent; however, there is much variance around this value.[8] In our baseline model, we assume that reform is equally likely at any date within the 75 year horizon of the model.[9]

The *current* Social Security policy (τ_1, b_1) in the model is parameterized to match the current policy in the US. Consistent with our modeling in earlier sections, we focus only on retirement insurance (the Old Age and Survivors, or OASI, program) and ignore disability insurance. The OASI payroll tax rate (combined employer and employee shares) is $\tau_1 = 10.6\%$. Benefits b_1 are chosen to match observed replacement rates for various income groups.

Social Security benefits are based on an individual's Average Indexed Monthly Earnings (AIME), calculated as average monthly earnings, indexed for economy-wide wage growth, over the highest 35 years of the individual's career. A progressive benefit formula is applied to AIME to arrive at an individual's Primary Insurance Amount (PIA), the monthly benefit payable if benefits are claimed at normal retirement age. Claiming before normal retirement age - for example, at age 65, as we assume in our model - results in an actuarial reduction to benefits. The progressive benefit formula implies that the replacement rate - the ratio of monthly benefits to AIME - falls with AIME.

The Social Security Trustees Report publishes replacement rates for several stylized workers, each earning a fixed multiple of the economy-wide average wage throughout their career. According to the 2013 Trustees Report, the very low income group, which earns 25 percent of the economy-wide average wage, receives a replacement rate of 67.5 percent of AIME if benefits are claimed at

[8] The Survey of Economic Expectations also elicits information on household expectations about future social security benefits (Dominitz et al. (2003), Manski (2004)). While this survey does document substantial uncertainty, it does not specifically measure uncertainty about the timing of reform. We are likely understating the costs of uncertainty because we are assuming that social security will continue to exist no matter what, whereas in reality a large portion of young households are not even confident of that (Dominitz et al. (2003)).

[9] There are two other assumptions that seem natural, which we consider for sensitivity analysis. First, the 75 year window over possible reform dates is arbitrary. Therefore, we consider shorter and longer horizons for reform. Second, it is plausible that as the trust fund comes closer to bankruptcy, the political pressure for reform will increase. For this reason we will also consider scenarios where the likelihood of reform rises as the trust fund exhaustion date approaches.

age 65. The low-income group, which earns 45 percent of the economy-wide average wage, has a replacement rate of 49.0 percent of AIME. The medium income group, which earns the economy-wide average wage, receives a replacement rate of 36.4 percent of AIME. The high income group, which earns 1.6 times the average wage, receives a replacement rate of 30.1 percent of AIME. Finally, workers who earn the maximum taxable amount in each year of their career receive a replacement rate of 24.0% of AIME. These replacement rates apply to the year 2055, when the 1990 cohort turns 65. To compute pre-reform benefits, b_1, we apply these replacement rates to the AIME for the normalized life-cycle income profile in our model, which is 0.92927.[10]

For the policy experiments considered in the paper we assume that the reform undertaken will balance the Social Security budget over the infinite horizon.[11] According to Gokhale (2013) the unfunded liabilities of the OASI program over the infinite horizon would require a permanent tax increase of 3.1 percent of payroll (assuming no behavioral response to the tax increase). Alternatively, all current and future benefits could be cut by 22.9 percent. As an initial pass, we consider both of these possible reforms:

- **Full benefit cut**. An across-the-board reduction in benefits by 22.9% for all current and future retirees (no exemption for current retirees) and no change in taxes. Hence, the new policy is $(\tau_2, b_2) = (\tau_1, b_1 \times (1 - 22.9\%))$.

- **Full tax increase**. An across-the-board increase in the OASI tax rate by 3.1 percentage points for all taxpayers regardless of age and no change in benefits. Hence, $(\tau_2, b_2) = (\tau_1 + 3.1\%, b_1)$.

We consider each of these reforms separately and assume that individuals know the reform that will occur, but they are uncertain about the timing. The results from these two scenarios will clarify the intuition of how timing uncertainty influences individual consumption and savings decisions. We will later consider the case where individuals are also uncertain about which of the two reforms will occur.

[10] In computing AIME for our life cycle model, we disregard wage indexation because the Gourinchas and Parker (2002) real income profiles are already adjusts not only for price inflation, but also for cohort and time effects.

[11] A partial reform may extend the horizon of the timing uncertainty, leaving the welfare costs that we consider in the paper.

5. Results

In order to gain intuition about the effects of uncertainty, we plot the results for the case of a benefit cut. Figure 1 shows three consumption profiles over the life cycle for the full benefit cut experiment, for an individual with average earnings: the consumption path during the first stage c_1^*; a hypothetical consumption path during stage two, c_2^*, conditional on reform at $t_1 = 0.4$; and the consumption profile from a world with no risk where the individual gets her expected lifetime income, c^{NR}, as a reference point. The path the individual actually follows is c_1^* up to the stochastic date of reform and then consumption drops down to c_2^*. The figure shows that in the case of reform at time $t_1 = 0.4$ there is a sizable drop in consumption at the date of reform that then approaches the no risk consumption path as time continues. An important feature of the experiment to emphasize is that the individual is still subject to a benefit cut even after the retirement date, so the first stage consumption path takes this possibility into account. While not shown, the tax increase reform displays smaller drops in consumption that only occur up until the date of retirement and no drop if reform occurs after that date.

Our first set of results corresponds to the welfare loss for an individual with average earnings. We find that the magnitude of the welfare loss is moderate for this individual: 0.044% of lifetime consumption for the case of a benefit cut. We also find that uncertainty about the timing of tax reform is on average less costly than uncertainty about the timing of benefit reform. The former is slightly less than *half* as costly as the latter.[12]

Why does the welfare cost depend so much on the structure of reform? Uncertainty about the future tax rate is less costly because the reform only impacts individuals during their working lives. Therefore, the later that reform occurs implies that individuals face less of the tax burden. Moreover, once they are retired reform does not hit them at all, providing a natural exemption for retired workers who have less income available in each period to enable them to change their consumption and savings plans. On the other hand, benefit cuts only start to impact individuals once they hit retirement meaning that later reform dates still have large impacts on individual consumption. In other words, uncertainty about taxes ceases to be costly to the individual once he

[12] While these numbers seem small, the fact that Social Security only taxes 10.6 percent of labor income implies that it is only a limited part of individuals overall lifetime consumption.

retires and is done paying taxes, whereas uncertainty about benefits never ceases to be a problem since he collects benefits until death.

We also find important distributional effects as we look beyond the average individual. Table 1 reports welfare losses associated with uncertainty about the timing of reform for different income levels (each relative to the average individual in the benefit reform scenario). Due to the progressively of benefits, uncertainty about the timing of a benefit cut is more harmful to low income individuals than to high income individuals. For example, Table 1 shows that the very low income group will experience a welfare loss that is over 5 times larger than what is experienced by the highest income group that maximizes their social security contribution in each year of work. This asymmetry occurs because social security benefits are a larger share of total retirement income for the poor than for the rich, and uncertainty over something important is naturally going to be costly.

But this distributional effect reverses its sign for the case of uncertainty about the timing of a tax increase. Essentially, now the progressivity argument works in the opposite way. Even though all income groups pay the same tax rate, uncertainty about the tax rate is more costly for the rich because social security benefits are smaller for them (relative to their wage) and hence they face uncertainty about a larger portion of their wealth than do the poor.

Table 1. Welfare Loss from Uncertainty about the Timing of Reform:
Baseline Parameters

Income Level	Repl. Rate	Full Benefit Reform (22.9% benefit cut)	Full Tax Reform (3.1 ppt increase)
very low	67.5%	2.78	0.36
low	49.0%	1.65	0.40
average	36.4%	**1.00***	0.44
high	30.1%	0.72	0.45
max	24.0%	0.49	0.47

*All other welfare costs are relative to this case.

6. Two Layers of Uncertainty: Timing and Structure of Reform

We now assume that both the timing and structure of social security reform are uncertain. Let $\tilde{\tau}_2$ be the new tax rate that would be sufficient to balance the budget without any reduction in benefits, and likewise let $\tilde{b}_2$ be the new benefit level that would balance the budget without a tax increase. Of course, $\tilde{\tau}_2 > \tau_1$ and $\tilde{b}_2 < b_1$. But the new policy that the government actually chooses (τ_2, b_2) is an uncertain, linear combination of these extremes. We will express the new tax policy as a function of a random variable α (with density $\theta(\alpha)$ and sample space $[0, 1]$):

$$\tau_2(\alpha) = \tau_1 + \alpha(\tilde{\tau}_2 - \tau_1), \tag{35}$$

$$b_2(\alpha) = b_1 - (1-\alpha)(b_1 - \tilde{b}_2), \tag{36}$$

and

$$y_2(t|\alpha) = \begin{cases} (1-\tau_2(\alpha))w(t), \text{ for } t \in [0, t_R], \\ b_2(\alpha), \text{ for } t \in [t_R, T]. \end{cases} \tag{37}$$

Using Theorem 2, the post-reform consumption path is

$$c_2^*(t|t_1, k(t_1), \alpha) = \frac{k(t_1) + \int_{t_1}^T e^{-r(v-t_1)} y_2(v|\alpha)dv}{\int_{t_1}^T e^{-r(v-t_1)+(r-\rho)v/\sigma}\Psi(v)^{1/\sigma}dv} e^{(r-\rho)t/\sigma}\Psi(t)^{1/\sigma}, \text{ for } t \in [t_1, T], \tag{38}$$

and the pre-reform solution Euler equation is

$$\begin{aligned} \frac{dc(t)}{dt} &= \left(\frac{c(t)^{\sigma+1}}{\Psi(t)} \int_0^1 \theta(\alpha) \left[\frac{k(t) + \int_t^T e^{-r(v-t)} y_2(v|\alpha)dv}{\int_t^T e^{-r(v-t)+(r-\rho)v/\sigma}\Psi(v)^{1/\sigma}dv} \right]^{-\sigma} e^{(\rho-r)t} d\alpha - c(t) \right) \\ &\quad \times \left[\frac{\sigma}{\phi(t)} \int_t^T \phi(t_1)dt_1 \right]^{-1} \\ &\quad + \left[\frac{d\Psi(t)}{dt} \frac{1}{\Psi(t)} + r - \rho \right] \frac{c(t)}{\sigma}. \end{aligned} \tag{39}$$

Using this Euler equation, together with the law of motion and boundary conditions for the savings account, we can compute the stage-one solution $(c_1^*(t), k_1^*(t))_{t\in[0,T]}$.

Finally, for welfare comparisons, the no risk benchmark is

$$c^{NR}(t) = \frac{\int_0^1 \int_0^T \theta(\alpha)\phi(t_1)Y(t_1|\alpha)dt_1 d\alpha}{\int_0^T e^{-rv+(r-\rho)v/\sigma}\Psi(v)^{1/\sigma}dv} e^{(r-\rho)t/\sigma}\Psi(t)^{1/\sigma}, \text{ for } t \in [0,T], \tag{40}$$

where $Y(t_1|\alpha) \equiv \int_0^{t_1} e^{-rv}y_1(v)dv + \int_{t_1}^T e^{-rv}y_2(v|\alpha)dv$, and the welfare cost of reform uncertainty Δ solves the following equation

$$\begin{aligned} &\int_0^T e^{-\rho t}\Psi(t)\frac{[c^{NR}(t)(1-\Delta)]^{1-\sigma}}{1-\sigma}dt \qquad (41) \\ &= \int_0^1 \int_0^T \theta(\alpha)\phi(t_1)\left(\int_0^{t_1} e^{-\rho t}\Psi(t)\frac{c_1^*(t)^{1-\sigma}}{1-\sigma}dt + \int_{t_1}^T e^{-\rho t}\Psi(t)\frac{c_2^*(t|t_1,k_1^*(t_1),\alpha)^{1-\sigma}}{1-\sigma}dt\right)dt_1 d\alpha. \end{aligned}$$

In the absence of reliable data on expectations about the structure of future reform, we will assume $\theta(\alpha)$ is the uniform density. Table 2 documents the welfare costs of double uncertainty for various income levels, each relative to the average earner in the full benefit reform scenario (in which case $\Delta = 0.034\%$). Two patterns emerge in Table 2. First, double uncertainty tends to produce welfare losses that are between the extremes of full benefit reform and full tax reform. Second, double uncertainty hurts the poor more than the rich (by a factor of more than 3).

Table 2. Welfare Loss from Uncertainty about Timing & Structure of Reform:

Baseline Parameters

Income Level	Repl. Rate	Full Benefit Reform (22.9% benefit cut)	Full Tax Reform (3.1 ppt increase)	Double Uncertainty
very low	67.5%	2.78	0.36	1.88
low	49.0%	1.65	0.40	1.16
average	36.4%	**1.00***	0.44	0.76
high	30.1%	0.72	0.45	0.60
max	24.0%	0.49	0.47	0.46

*All other welfare costs are relative to this case.

7. Robustness

While the uniform density over reform dates allows us to be agnostic about when reform will occur, it may be that political pressure for reform will mount as the trust fund runs out of money by 2033. We therefore consider a more flexible, two-parameter density

$$\phi(t_1) = \left[\int_0^1 e^{-\mu(\gamma t_1 - 1)^2} dt_1\right]^{-1} \times e^{-\mu(\gamma t_1 - 1)^2}. \tag{42}$$

Note that $\int_0^1 \phi(t_1)dt_1 = 1$, $mode = \arg\max \phi(t_1) = \gamma^{-1}$, and the parameter μ is inversely related to the spread (uniform density is a special case with $\mu = 0$). We set the mode at 2033 (model time 0.24), which implies $\gamma = 4.167$, and we set $\mu = 1.0$ as an example. These assumptions give a bell-shaped density that is skewed heavily to the left: there is a 95% chance that reform will occur before the individual reaches retirement.

For an individual with average earnings, the magnitude of the welfare loss is now much smaller for the case of uncertainty about the timing of a benefit cut ($\Delta = 0.004\%$). In Table 3 all other welfare costs are relative to this one. Notice that the distributional effects within a given reform are qualitatively similar to the baseline parameterization. That is, the poor are hit harder in the case of uncertainty over the timing of a benefit cut while the rich are hit harder in the case of uncertainy over the timing of a tax increase. However, because the new density puts most the mass during the working period, uncertainty about the timing of tax reform is far more costly than uncertainty about the timing of benefit reform (unlike in the baseline parameterization). If the individual is living in the benefit-reform world, he actually faces very little uncertainty about his lifetime wealth because he is nearly certain that benefits will be cut before he retires. Finally, as with the baseline density, double uncertainty continues to hit the poor very hard relative to the rich (again, by more than a factor of 3).

Table 3. Welfare Loss from Uncertainty about Timing & Structure of Reform:
Alternative Density Function

Income Level	Repl. Rate	Full Benefit Reform (22.9% benefit cut)	Full Tax Reform (3.1 ppt increase)	Double Uncertainty
very low	67.5%	1.36	3.37	8.98
low	49.0%	1.12	3.82	5.28
average	36.4%	**1.00***	4.17	3.49
high	30.1%	0.95	4.35	2.88
max	24.0%	0.92	4.54	2.49

*All other welfare costs are relative to this case.

8. Conclusion

In this paper we attempt to understand the welfare consequences of uncertainty over the timing and structure of social security reform. Household decision-making in the face of reform uncertainty can be modeled as a stochastic two-stage optimal control problem. We have formalized the tools that are required for studying this issue in a continuous-time setting.

We have paid special attention to how these welfare costs are distributed across income groups. The poor are hurt more than the rich when there is uncertainty over the timing of a benefit cut, while the rich are hurt more than the poor if there is uncertainty over the timing a tax increase.

Appendices

Appendix A provides a detailed proof of Theorem 1. Appendix B provides a full derivation of the solution to the household's optimization problem.

Appendix A: Proof of Theorem 1

Step 1. This step of the backward induction procedure is a completely standard Pontryagin problem and needs very little justification. The optimal control and state paths after the switch is realized must solve a standard deterministic control problem. We denote the solution to this subproblem $(u_2^*(t|t_1, x(t_1)), x_2^*(t|t_1, x(t_1)))_{t\in[t_1,T]}$, where the extra notation is meant to convey the dependence of the solution on the switch date t_1 and on the state variable at that date $x(t_1)$. The last part of Step 1 is strictly for convenience: we take this solution and change the time dummy t to z, and change the switch point t_1 to t and write $(u_2^*(z|t, x(t)), x_2^*(z|t, x(t)))_{z\in[t,T]}$. Thus we have the optimal control and state paths for all points in time z greater than switch point t. This change of dummies is innocuous but proves to be very helpful below in the process of linking the subproblems together though the continuation functional in the next step.

Step 2. This step requires a little more explanation. The purpose of this step is to find the optimal paths for the control and state before the realization of the switch. Hence, using the solution from Step 1, the objective functional is

$$\max_{u(t)_{t\in[0,T]}} : J_1 = \mathbb{E}\left[\int_0^{t_1} f_1(t, u(t), x(t))dt + \int_{t_1}^{T} f_2(t, u_2^*(t|t_1, x(t_1)), x_2^*(t|t_1, x(t_1)))dt\right] \tag{A1}$$

or

$$\max_{u(t)_{t\in[0,T]}} : J_1 = \int_0^T \int_0^{t_1} \phi(t_1) f_1(t, u(t), x(t))dtdt_1 + \int_0^T \phi(t_1)\mathcal{S}(t_1, x(t_1))dt_1, \tag{A2}$$

where $\mathcal{S}(t_1, x(t_1))$ is the continuation value or continuation function,

$$\mathcal{S}(t_1, x(t_1)) = \int_{t_1}^{T} f_2(t, u_2^*(t|t_1, x(t_1)), x_2^*(t|t_1, x(t_1)))dt, \tag{A3}$$

and likewise $\int_0^T \phi(t_1)\mathcal{S}(t_1, x(t_1))dt_1$ is the continuation **functional.** The constraints are

$$\frac{dx(t)}{dt} = g_1(t, u(t), x(t)), \text{ for } t \in [0, T], \tag{A4}$$

$$x(0) = x_0,\ x(T) = x_T. \tag{A5}$$

Note that the control and state variables are defined over the entire planning interval because the switch could happen at any time on this interval, and hence the pre-switch problem amounts to choosing a path for these variables over the full interval. The marginal valuation of the state variable reflects the fact that the switch could happen at any moment in time, and hence the entire timepath of the state variable is relevant in determining the continuation value during the second stage. This makes matters more complicated than in the standard two-stage model with a deterministic switch, because there the costate variables from the two stages are simply equated at the switch point to ensure that the value of finishing the first stage with an extra unit of the state variable is equal to the value of starting the second stage with another unit. Whereas here the two stages are linked together through a continuation functional. The solution to the above problem is the one that will be followed up to the random switch point.

This appears to be a non-standard control problem, but with some algebra we can convert it into a standard one for which the standard Maximum Principle applies. To make progress, let's change the dummy of integration in the second integral in J_1 from t_1 to t, and also change the dummy in the continuation functional from t to z. Now we restate the objective functional as

$$\max_{u(t)_{t\in[0,T]}} : J_1 = \int_0^T \int_0^{t_1} \phi(t_1) f_1(t, u(t), x(t))dt dt_1 + \int_0^T \phi(t)\mathcal{S}(t, x(t))dt, \tag{A6}$$

where

$$\mathcal{S}(t, x(t)) = \int_t^T f_2(z, u_2^*(z|t, x(t)), x_2^*(z|t, x(t)))dz. \tag{A7}$$

This innocuous change of variables is helpful because it allows us to write the continuation function $\mathcal{S}$ as a function of $x(t)$. In doing so, $u_2^*(z|t, x(t))$ and $x_2^*(z|t, x(t))$ now take on the interpretation of the optimal control and state variables for all points in time z that are beyond the switch date t and conditional on $x(t)$.

Our problem can be yet simplified as follows. First change the order of integration in the first term in J_1 by integrating by parts, where $\Phi(t_1)$ is an antiderivative of $\phi(t_1)$, $\Phi'(t_1) = \phi(t_1)$,

$$\begin{aligned}\int_0^T \int_0^{t_1} \phi(t_1) f_1(t, u(t), x(t)) dt dt_1 &= \left[\Phi(t_1)\left(\int_0^{t_1} f_1(t, u(t), x(t)) dt\right)\right]_0^T \\ &\quad - \int_0^T \Phi(t_1) f_1(t_1, u(t_1), x(t_1)) dt_1 \\ &= \Phi(T)\left(\int_0^T f_1(t, u(t), x(t)) dt\right) \\ &\quad - \int_0^T \Phi(t_1) f_1(t_1, u(t_1), x(t_1)) dt_1. \end{aligned} \tag{A8}$$

Change the dummy of integration in the last integral on the right hand side,

$$\begin{aligned}\int_0^T \int_0^{t_1} \phi(t_1) f_1(t, u(t), x(t)) dt dt_1 &= \Phi(T)\left(\int_0^T f_1(t, u(t), x(t)) dt\right) \\ &\quad - \int_0^T \Phi(t) f_1(t, u(t), x(t)) dt. \end{aligned} \tag{A9}$$

Continuing on and using the Fundamental Theorem of Calculus

$$\int_0^T \int_0^{t_1} \phi(t_1) f_1(t, u(t), x(t)) dt dt_1 = \int_0^T \left[\int_t^T \phi(t_1) dt_1\right] f_1(t, u(t), x(t)) dt. \tag{A10}$$

We now see that a more manageable Pontryagin problem has emerged. Let us now restate our problem one last time

$$\max_{u(t)_{t\in[0,T]}} : J_1 = \int_0^T \left\{\left[\int_t^T \phi(t_1) dt_1\right] f_1(t, u(t), x(t)) + \phi(t)\mathcal{S}(t, x(t))\right\} dt, \tag{A11}$$

subject to

$$\mathcal{S}(t, x(t)) = \int_t^T f_2(z, u_2^*(z|t, x(t)), x_2^*(z|t, x(t))) dz, \tag{A12}$$

$$\frac{dx(t)}{dt} = g_1(t, u(t), x(t)), \text{ for } t \in [0, T], \tag{A13}$$

$$x(0) = x_0, \ x(T) = x_T. \tag{A14}$$

This reformulated objective functional has an intuitive interpretation. The first term in the integrand gives the payoff of $u(t)$ and $x(t)$ through the function $f_1(t, u(t), x(t))$, weighted by the probability that the random switch will occur sometime after t. The second term gives the payoff of holding $x(t)$ through the continuation value $\mathcal{S}(t, x(t))$, weighted by the density function $\phi(t)$. Thus, the first payoff term f_1 is weighted by one minus the c.d.f. because this payoff is relevant as long as the switch comes later than t, whereas the second payoff term $\mathcal{S}$ is weighted by the p.d.f. because this payoff is relevant only at the switch point. The standard Maximum Principle can be applied to this reformulated subproblem. We denote the solution to this subproblem $(u_1^*(t), x_1^*(t))_{t\in[0,T]}$. This is the solution path for all t before the realization of the random switch. Note that the switch could happen at any point in time. This is why this subproblem is defined over the entire planning interval.

Appendix B. Derivation of Solution to Household Problem

Using Theorem 1 as our guide, we can solve the stochastic two-stage household problem recursively by breaking it into two Pontryagin subproblems as follows.

Step 1. Solve the post-reform ($t = t_1$) subproblem:

We first solve the Pontryagin subproblem corresponding to the moment that reform occurs. This is a deterministic, fixed endpoint control problem

$$\max_{c(t)_{t\in[t_1,T]}} : J_2 = \int_{t_1}^{T} e^{-\rho t}\Psi(t)\frac{c(t)^{1-\sigma}}{1-\sigma}dt, \tag{B1}$$

subject to

$$\frac{dk(t)}{dt} = rk(t) + y_2(t) - c(t), \text{ for } t \in [t_1, T], \tag{B2}$$

$$t_1 \text{ given}, k(t_1) \text{ given}, k(T) = 0. \tag{B3}$$

Form the Hamiltonian $\mathcal{H}_2$ with multiplier $\lambda_2(t)_{t\in[t_1,T]}$ and compute the necessary conditions

$$\mathcal{H}_2 = e^{-\rho t}\Psi(t)\frac{c(t)^{1-\sigma}}{1-\sigma} + \lambda_2(t)[rk(t) + y_2(t) - c(t)], \tag{B4}$$

$$\frac{\partial\mathcal{H}_2}{\partial c(t)} = e^{-\rho t}\Psi(t)c(t)^{-\sigma} - \lambda_2(t) = 0, \text{ for } t \in [t_1, T], \tag{B5}$$

$$\frac{d\lambda_2(t)}{dt} = -\frac{\partial\mathcal{H}_2}{\partial k(t)} = -r\lambda_2(t), \text{ for } t \in [t_1, T]. \tag{B6}$$

Rewrite the costate equation

$$\lambda_2(t) = \lambda_2(t_1)e^{-r(t-t_1)}, \tag{B7}$$

and collapse the necessary conditions into a single equation

$$e^{-\rho t}\Psi(t)c(t)^{-\sigma} = \lambda_2(t_1)e^{-r(t-t_1)}. \tag{B8}$$

Solve for $c(t)$

$$c(t) = \lambda_2(t_1)^{-1/\sigma}e^{[(r-\rho)t-rt_1]/\sigma}\Psi(t)^{1/\sigma}. \tag{B9}$$

Solve differential equation (B2) using the boundary conditions in (B3)

$$k(t_1) + \int_{t_1}^{T} y_2(v)e^{-r(v-t_1)}dv = \int_{t_1}^{T} c(v)e^{-r(v-t_1)}dv. \tag{B10}$$

Insert (B9) into (B10)

$$k(t_1) + \int_{t_1}^{T} y_2(v)e^{-r(v-t_1)}dv = \int_{t_1}^{T} \lambda_2(t_1)^{-1/\sigma}e^{-r(v-t_1)+[(r-\rho)v-rt_1]/\sigma}\Psi(v)^{1/\sigma}dv, \tag{B11}$$

and solve for the constant

$$\lambda_2(t_1)^{-1/\sigma} = \frac{k(t_1) + \int_{t_1}^{T} y_2(v)e^{-r(v-t_1)}dv}{\int_{t_1}^{T} e^{-r(v-t_1)+[(r-\rho)v-rt_1]/\sigma}\Psi(v)^{1/\sigma}dv}. \tag{B12}$$

Insert this into (B9) to obtain the solution consumption path

$$c_2^*(t|t_1, k(t_1)) = \frac{k(t_1) + \int_{t_1}^{T} e^{-r(v-t_1)}y_2(v)dv}{\int_{t_1}^{T} e^{-r(v-t_1)+(r-\rho)v/\sigma}\Psi(v)^{1/\sigma}dv}e^{(r-\rho)t/\sigma}\Psi(t)^{1/\sigma}, \text{ for } t \in [t_1, T]. \tag{B13}$$

This is the optimal consumption path after the reform shock has hit.

Anticipating the method for solving the pre-reform subproblem in the next step of Theorem 1, we will need to change the time dummies: now think of t as the reform date and z as any time after the reform date. Thus, rewrite the solution as

$$c_2^*(z|t, k(t)) = \frac{k(t) + \int_{t}^{T} e^{-r(v-t)}y_2(v)dv}{\int_{t}^{T} e^{-r(v-t)+(r-\rho)v/\sigma}\Psi(v)^{1/\sigma}dv}e^{(r-\rho)z/\sigma}\Psi(z)^{1/\sigma}, \text{ for } z \in [t, T]. \tag{B14}$$

Step 2. Solve the pre-reform ($t = 0$) subproblem:

Next we solve the $t = 0$ subproblem as in Theorem 1,

$$\max_{c(t)_{t\in[0,T]}} : J_1 = \int_0^T \left\{ \left[\int_t^T \phi(t_1)dt_1 \right] e^{-\rho t}\Psi(t)\frac{c(t)^{1-\sigma}}{1-\sigma} + \phi(t)\mathcal{S}(t, k(t)) \right\} dt, \tag{B15}$$

subject to

$$\begin{aligned} \mathcal{S}(t, k(t)) &= \int_t^T e^{-\rho z}\Psi(z)\frac{c_2^*(z|t, k(t))^{1-\sigma}}{1-\sigma}dz \\ &= \frac{1}{1-\sigma}\left[\frac{k(t) + \int_t^T e^{-r(v-t)}y_2(v)dv}{\int_t^T e^{-r(v-t)+(r-\rho)v/\sigma}\Psi(v)^{1/\sigma}dv}\right]^{1-\sigma} \int_t^T e^{[r(1-\sigma)-\rho]z/\sigma}\Psi(z)^{1/\sigma}dz, \end{aligned} \tag{B16}$$

$$\frac{dk(t)}{dt} = rk(t) + y_1(t) - c(t), \text{ for } t \in [0, T], \tag{B17}$$

$$k(0) = 0,\ k(T) = 0. \tag{B18}$$

Form the Hamiltonian $\mathcal{H}_1$ with multiplier $\lambda_1(t)_{t\in[0,T]}$ and compute the necessary conditions[13]

$$\mathcal{H}_1 = \left[\int_t^T \phi(t_1)dt_1 \right] e^{-\rho t}\Psi(t)\frac{c(t)^{1-\sigma}}{1-\sigma} + \phi(t)\mathcal{S}(t, k(t)) + \lambda_1(t)[rk(t) + y_1(t) - c(t)], \tag{B19}$$

$$\frac{\partial \mathcal{H}_1}{\partial c(t)} = \left[\int_t^T \phi(t_1)dt_1 \right] e^{-\rho t}\Psi(t)c(t)^{-\sigma} - \lambda_1(t) = 0, \text{ for } t \in [0, T], \tag{B20}$$

$$\frac{d\lambda_1(t)}{dt} = -\frac{\partial \mathcal{H}_1}{\partial k(t)} = -\phi(t)\left[\frac{k(t) + \int_t^T e^{-r(v-t)}y_2(v)dv}{\int_t^T e^{-r(v-t)+(r-\rho)v/\sigma}\Psi(v)^{1/\sigma}dv}\right]^{-\sigma} e^{-rt} - r\lambda_1(t), \text{ for } t \in [0, T]. \tag{B21}$$

Differentiate (B20) with respect to t

$$\begin{aligned} 0 &= -\phi(t)e^{-\rho t}\Psi(t)c(t)^{-\sigma} + \left[\int_t^T \phi(t_1)dt_1 \right]\left[\frac{d\Psi(t)}{dt}e^{-\rho t} - \rho\Psi(t)e^{-\rho t} \right] c(t)^{-\sigma} \\ &\quad - \left[\int_t^T \phi(t_1)dt_1 \right] \sigma e^{-\rho t}\Psi(t)c(t)^{-\sigma-1}\frac{dc(t)}{dt} - \frac{d\lambda_1(t)}{dt}. \end{aligned} \tag{B22}$$

[13]The necessary conditions are also sufficient because the integrand of J_1 is concave in $c(t)$ and $k(t)$ (see Corollary 1).

Insert (B20) into (B21)

$$\frac{d\lambda_1(t)}{dt} = -\phi(t)\left[\frac{k(t)+\int_t^T e^{-r(v-t)}y_2(v)dv}{\int_t^T e^{-r(v-t)+(r-\rho)v/\sigma}\Psi(v)^{1/\sigma}dv}\right]^{-\sigma} e^{-rt} - r\left[\int_t^T \phi(t_1)dt_1\right] e^{-\rho t}\Psi(t)c(t)^{-\sigma}, \quad \text{(B23)}$$

and then insert (B23) into (B22) to obtain the Euler equation

$$\begin{aligned}\frac{dc(t)}{dt} &= \left(\frac{c(t)^{\sigma+1}}{\Psi(t)}\left[\frac{k(t)+\int_t^T e^{-r(v-t)}y_2(v)dv}{\int_t^T e^{-r(v-t)+(r-\rho)v/\sigma}\Psi(v)^{1/\sigma}dv}\right]^{-\sigma} e^{(\rho-r)t} - c(t)\right) \times \left[\frac{\sigma}{\phi(t)}\int_t^T \phi(t_1)dt_1\right]^{-1} \\ &\quad + \left[\frac{d\Psi(t)}{dt}\frac{1}{\Psi(t)} + r - \rho\right]\frac{c(t)}{\sigma}. \qquad \text{(B24)}\end{aligned}$$

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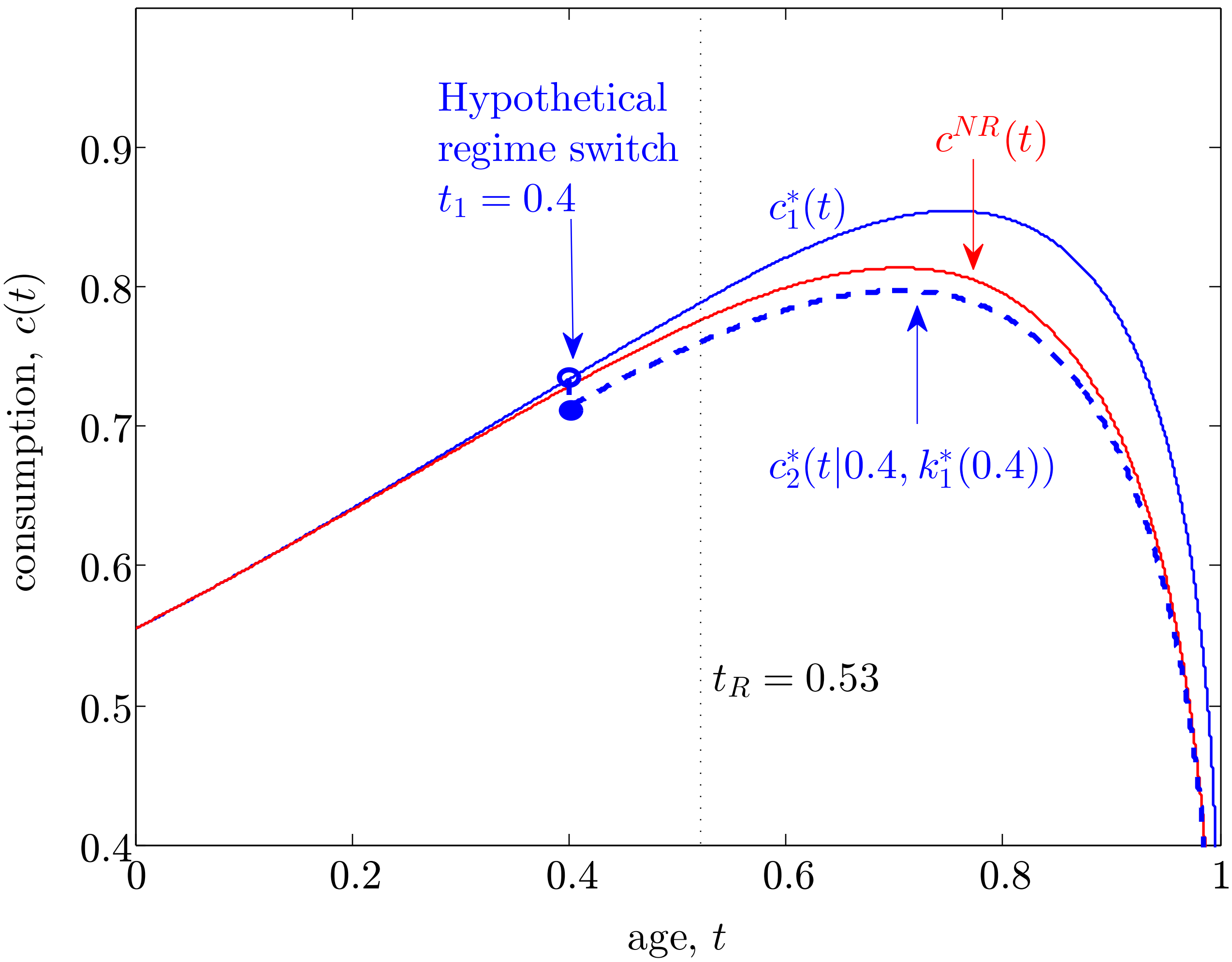
Figure 1. The Case of **Benefit Reform** with Stochastic Reform Date



Social security parameters: $\tau_1 = 0.106$, $\tau_2 = 0.106$, $b_1 = 0.322$, $b_2 = 0.248$.